Improved Exploration of Rectilinear Polygons *

Mikael Hammar[†] Bengt J. Nilsson[‡] Sven Schuierer[§]

Abstract

We prove a 5/3-competitive strategy for exploring a simple rectilinear polygon in the L_1 metric. This improves the previous factor two bound of Deng, Kameda and Papadimitriou.

1 Introduction

Searching in an environment is an important and well studied problem in robotics. In many realistic situations the robot does not possess complete knowledge about its environment, for instance, it may not have a map of its surroundings [1, 2, 5, 6, 7, 9, 8, 10, 11, 12, 13, 14].

The search of the robot can be viewed as an *on-line* problem since the robot's decisions about the search are based only on the part of its environment that it has seen so far. We use the framework of *competitive analysis* to measure the performance of an on-line search strategy S [15]. The *competitive ratio* of S is defined as the maximum of the ratio of the distance traveled by a robot using S to the optimal distance of the search.

In fact, these search problems come in basically two flavors. In some cases we want to search for a target in the environment [5, 6, 9, 10, 11, 12]. The target is usually modeled as a point in the environment that has to be reached by the robot. Another problem type is that of *exploring* an environment, i.e., without a priori knowledge of the environment let a robot construct a map of it [7, 8, 11].

We are interested in obtaining improved upper bounds for the competitive ratio of exploring a rectilinear polygon. The search is modeled by a path or closed tour followed by a point sized robot inside the polygon, given a starting point for the search. The only information that the robot has about its surrounding polygon is the part of the polygon that it has seen so far. Deng *et al.* [7] show a deterministic strategy having competitive ratio two for this problem if distance is measured according to the L_1-metric. Kleinberg [11] proves a lower bound of 5/4 for the competitive ratio of any deterministic strategy. We will show a strategy obtaining a competitive ratio of 5/3 for searching a rectilinear polygon in the L_1-metric.

*This research is supported by the DFG-Project "Diskrete Probleme", No. Ot 64/8-1.

[†]Department of Computer Science, Lund University, Box 118, S-221 00 Lund, Sweden.
email: `Mikael.Hammar@cs.lth.se`

[‡]Technology and Society, Malmö University College, S-205 06 Malmö, Sweden.
email: `Bengt.Nilsson@te.mah.se`

[§]Institut für Informatik, Am Flughafen 17, Geb. 051, D-79110 Freiburg, Germany.
email: `schuiere@informatik.uni-freiburg.de`

The paper is organized as follows. In the next section we present some definitions and preliminary results. In Section 3 we give an overview of the strategy by Deng *et al.* [7]. Section 4 contains an improved version giving a competitive ratio of 5/3.

2 Preliminaries

We will henceforth always measure distance according to the L_1 metric, i.e., the distance between two points p and q is defined by

$$||p, q|| = |p_x - q_x| + |p_y - q_y|,$$

where p_x and q_x are the x-coordinates of p and q and p_y and q_y are the y-coordinates. We define the x-distance between p and q to be $||p, q||_x = |p_x - q_x|$ and the y-distance to be $||p, q||_y = |p_y - q_y|$.

If C is a polygonal curve, then the length of C, denoted *length*(C), is defined as the sum of the distances between consecutive pairs of segment end points in C.

Let **P** be a simple rectilinear polygon. Two points in **P** are said to *see* each other, or be *visible* to each other, if the line segment connecting the points lies in **P**. For a point p in **P** we denote by $\mathbf{VP}(p)$ the set of points of **P** that are visible by the robot if it stands on position p. $\mathbf{VP}(p)$ is called the *visibility polygon* of p. If C is a curve in **P**, the visibility polygon $\mathbf{VP}(C)$ of C is defined by $\mathbf{VP}(C) = \bigcup_{p \in C} \mathbf{VP}(p)$.

Let p be a point somewhere inside **P**. A *watchman route* through p is defined to be a closed curve C that passes through p with $\mathbf{VP}(C) = \mathbf{P}$. The shortest watchman route through p is denoted by SWR_p. It can be shown that the shortest watchman route in a simple polygon is a closed polygonal curve [3, 4].

Since we are only interested in the L_1 length of a polygonal curve we can assume that the curve is *rectilinear*, that is, the segments of the curve are all axis parallel. Note that the shortest rectilinear watchman route through a point p is not necessarily unique.

For a point p in **P** we define four *quadrants* with respect to p. Those are the regions obtained by cutting **P** along the two maximal axis parallel line segments that pass through p. The four quadrants are denoted $\mathbf{Q1}(p)$, $\mathbf{Q2}(p)$, $\mathbf{Q3}(p)$, and $\mathbf{Q4}(p)$ in the order from the top right quadrant to the bottom right quadrant; see Figure 1(a).

Consider a reflex vertex of **P**. The two edges of **P** connecting at the reflex vertex can each be extended inside **P** until the extensions reach a boundary point. The segments thus constructed are called *extensions* and to each extension a direction is associated. The direction is the same as that of the collinear polygon edge as we follow the boundary of **P** in clockwise order; see Figure 1(b). We use the four compass directions *north*, *west*, *south*, and *east* to denote the direction of an extension. The reason for this somewhat obscure definition is hopefully clarified by the following lemma.

LEMMA 2.1 (CHIN, NTAFOS [3]) *A closed curve is a watchman route for* **P** *if and only if the curve has at least one point to the right of every extension of* **P**.

Our objective is thus to present a competitive on line strategy that enables a robot to follow a closed curve from the start point s in **P** and back to s with the curve being a watchman route for **P**.

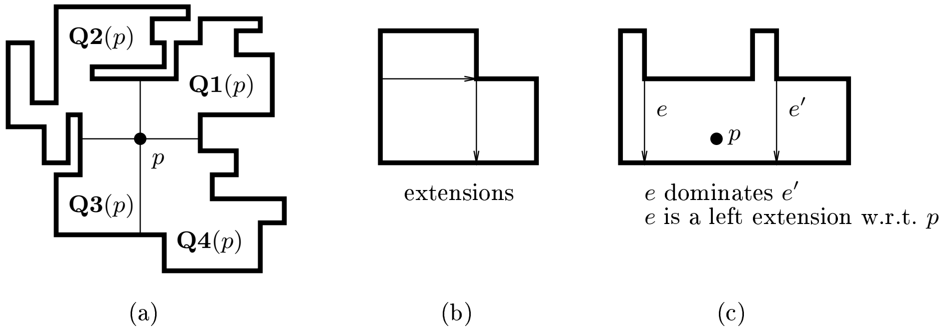


Figure 1: Illustrating definitions.

An extension e splits $\mathbf{P}$ into two sets $\mathbf{P}_l$ and $\mathbf{P}_r$ with $\mathbf{P}_l$ to the left of e and $\mathbf{P}_r$ to the right. We say a point p is *to the left of* e if p belongs to $\mathbf{P}_l$. *To the right* is defined analogously.

As a further definition we say that an extension e is *a left extension* with respect to a point p, if p lies to the left of e, and an extension e *dominates* another extension e', if all points of $\mathbf{P}$ to the right of e are also to the right of e'; see Figure 1(c). By Lemma 2.1 we are only interested in the extensions that are left extensions with respect to the starting point s since the other ones already have a point (the point s) to the right of them. So without loss of clarity when we mention extensions we will always mean extensions that are left extensions with respect to s.

As a prelude to the presentation of our strategy we give a short overview of previous strategies. In this way we can introduce concepts in their original setting that we will use extensively in the sequel.

3 An Overview of GO

Consider a rectilinear polygon $\mathbf{P}$ that is not a priori known to the robot. Let s be the robot's initial position inside $\mathbf{P}$.

The next concept we introduce is crucial to the workings of previously presented strategies and is going to be very important for our strategy as well. For the starting position s of the robot we associate a point f^0 on the boundary of $\mathbf{P}$ that is visible from s and call f^0 the *principal projection point* of s. For instance, we can choose f^0 to be the first point on the boundary that is hit by an upward ray starting at s. Assume now that the robot moves along a curve C from s to some point p. Let f be the end point that the robot has seen so far as we move clockwise from f^0 along the boundary of $\mathbf{P}$; see Figure 2. The point f is called the current *frontier*.

Let C be a polygonal curve starting at s. Formally a *frontier* f of C is a vertex of $\mathbf{VP}(C)$ adjacent to an edge e of $\mathbf{VP}(C)$ that is not an edge of $\mathbf{P}$. Extend e until it hits a point q on C and let v be the vertex of $\mathbf{P}$ that is first encountered as we move along the line segment $[q, f]$ from q to f. We denote the left extension with respect to s associated to the vertex v by $ext(f)$; see Figures 2(b) and (c).

Deng *et al.* [7] introduce an on-line strategy called *greedy-online*, GO for short, to explore a simple rectilinear polygon $\mathbf{P}$ in the L_1 metric. If the starting point s lies on the boundary of $\mathbf{P}$,

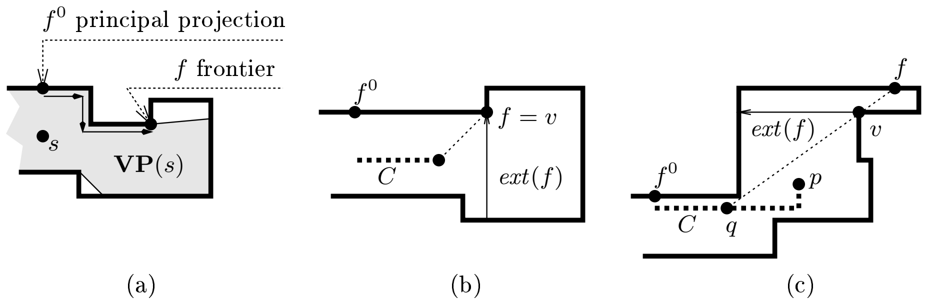


Figure 2: Illustrating definitions.

their strategy, we call it BGO, goes as follows.

Strategy BGO
1 Using s itself as principal projection point establish the first frontier f
2 $p := s$
3 **while** **P** is not completely seen **do**
3.1 Go to the point on $ext(f)$ closest to p, let this point be the new point p
3.2 Using the current point f on the boundary as principal projection point establish the new frontier f
endwhile
4 Take the shortest path back to s
End BGO

Deng *et al.* show that a robot using strategy BGO to explore a rectilinear polygon follows a tour with shortest length, i.e., BGO has competitive ratio one. They also present a similar strategy, called IGO, for the case when the starting point s lies in the interior of **P**. For IGO they show a competitive ratio of two, i.e., IGO specifies a tour that is at most twice as long as the shortest watchman route. IGO does the following.

Strategy IGO
1 From s shoot a ray upwards until a boundary point f^0 is hit, use f^0 as principal projection point and establish the first frontier f
2 $p := s$
3 **while** **P** is not completely seen **do**
3.1 Go to the point on $ext(f)$ closest to p, let this point be the new point p
3.2 Using the current point f on the boundary as principal projection point establish the new frontier f
endwhile
4 Take the shortest path back to s
End IGO

For an example run of strategy IGO see Figure 3. The points f^i, for $1 \leq i \leq 19$, are the frontiers in the order that they are encountered along the boundary. The directed lines are the extensions associated to the frontiers.

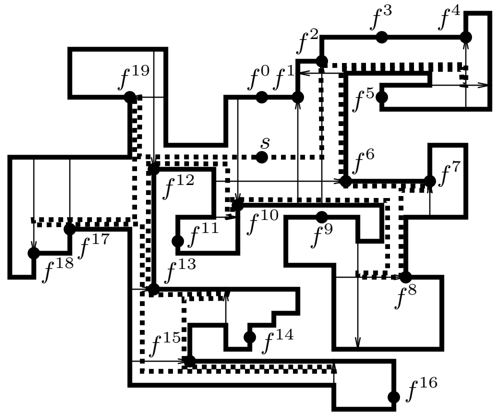


Figure 3: Example run of the greedy-online strategy.

As is easily seen, the two strategies differ only in the first step where in one case we establish the first frontier starting directly from s, whereas in the other case we shoot a ray in order to find the principal projection point and from there we establish the first frontier.

It is clear that BGO could just as well scan the boundary anti-clockwise instead of clockwise when establishing the frontiers and still have the same competitive ratio. Hence, BGO can be seen as two strategies, one scanning clockwise and the other anti-clockwise. We can therefore parameterize the two strategies so that BGO(p, *orient*) is the strategy beginning at some point p on the boundary and scanning with orientation *orient* where *orient* is either clockwise *cw* or anti-clockwise *aw*.

Similarly for IGO, we can not only choose to scan clockwise or anti-clockwise for the frontier but also choose to shoot the ray giving the first principal projection point in any of the four compass directions north, west, south, or east. Thus IGO in fact becomes eight different strategies that we can parameterize as IGO(p, *dir*, *orient*) and the parameter *dir* can be one of *north*, *south*, *west*, or *east*.

We further define partial versions of GO starting at boundary and interior points. Strategies PBGO(p, *orient*, *region*) and PIGO(p, *dir*, *orient*, *region*) apply GO until either the robot has explored all of *region* or the robot leaves the region *region*. The strategies return as result the position of the robot when it leaves *region* or when *region* has been explored. Note that PBGO(p, *orient*, $\mathbf{P}$) and PIGO(p, *dir*, *orient*, $\mathbf{P}$) are the same strategies as BGO(p, *orient*) and IGO(p, *dir*, *orient*) respectively except that they do not move back to p when all of $\mathbf{P}$ has been seen.

4 The Strategy DGO

We present a new strategy *double greedy online* (DGO) that starts off by exploring the first quadrant of s, $\mathbf{Q1}(s)$, without using up too much distance. We assume that s lies in the interior of $\mathbf{P}$ since otherwise we can use BGO and achieve an optimal route. The strategy uses two frontier points simultaneously (hence, the name) to improve the competitive ratio. The *upper frontier* f_u is the

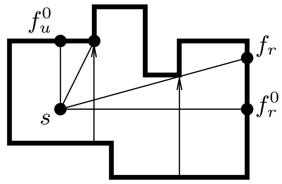


Figure 4: Upper and right frontiers together with their extensions.

same as the one used in GO, i.e., we shoot a ray straight up to get the upper principal projection point f_u^0 and scan in clockwise direction for f_u. The *right frontier* f_r is established by shooting a ray towards the right for the right principal projection point f_r^0 and then scan the boundary in anti-clockwise direction for f_r; see Figure 4. To each frontier point we associate a left extension $ext(f_u)$ and $ext(f_r)$ with respect to s in the same way as before.

The strategy DGO, presented in pseudo code on Page 7, makes use of five different substrategies: DGO-0, DGO-1, DGO-2, DGO-3, and DGO-4, that each take care of a specific case that can occur. Subsequently we will prove the correctness and competitive ratio for each of the substrategies.

We distinguish four classes of extensions. $\mathcal{A}$ is the class of extensions e whose defining edge is above e, $\mathcal{B}$ is the class of extensions e whose defining edge is below e. Similarly, $\mathcal{L}$ is the class of extensions e whose defining edge is to the left of e, and $\mathcal{R}$ is the class of extensions e whose defining edge is to the right of e. For conciseness, we use $\mathcal{C}_1\mathcal{C}_2$ as a shorthand for the cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$ of the two classes $\mathcal{C}_1$ and $\mathcal{C}_2$.

We will ensure that whenever the strategy performs one of the substrategies this is the last time that the outermost while-loop is executed. Hence, the loop is repeated only when the strategy does not enter any of the specified substrategies. The loop will lead the strategy to follow an x-y-monotone path inside $\mathbf{Q1}(s)$ since we will ensure that $ext(f_u)$ and $ext(f_r)$ always lie either above or to the right of the point p, the current position of the robot, in these cases. The strategy additionally maintains the invariant during the while-loop that the boundary parts of $\mathbf{Q1}(s)$ to the left of the x-coordinate of p and below the y-coordinate of p have been seen.

We will present each of the chosen substrategies in sequence and for each of them prove that if DGO executes the substrategy then the competitive ratio of DGO is bounded by 5/3. Let FR_s be the closed route followed by strategy DGO starting at an interior point s. Let $FR_s(p, q, orient)$ denote the subpath of FR_s followed in direction *orient* from point p to point q, where *orient* can either be *cw* (clockwise) or *aw* (anti-clockwise). Similarly, we define the subpath $SWR_s(p, q, orient)$ of SWR_s. We denote by $SP(p, q)$ a shortest rectilinear path from p to q inside $\mathbf{P}$.

We begin by establishing two simple but useful lemmas.

LEMMA 4.1 *If t is a point on some tour SWR_s, then*

$$length(SWR_t) \leq length(SWR_s).$$

PROOF: Since SWR_s passes through t, the route is a watchman route through t. But since SWR_t is the shortest watchman route through t, the lemma follows. □

If f is a frontier point, then let $reg(f)$ denote the region to the right of $ext(f)$.

Strategy DGO
Establish the upper and right principal projection points $f_u[s]$ and $f_r[s]$
while $\mathbf{Q1}(s)$ is not completely seen **do**
Obtain the upper and right frontiers, f_u and f_r
Case 1: $reg(f_u) \subseteq reg(f_r)$
Case 1.1: $ext(f_r) \in \mathcal{A} \cup \mathcal{R}$
Go to the closest point on $ext(f_r)$
Case 1.2: $ext(f_r) \in \mathcal{B} \cup \mathcal{L}$
Apply substrategy DGO-1

Case 2: $reg(f_r) \subseteq reg(f_u)$
Case 2.1: $ext(f_u) \in \mathcal{A} \cup \mathcal{R}$
Go to the closest point on $ext(f_u)$
Case 2.2: $ext(f_u) \in \mathcal{B} \cup \mathcal{L}$
Apply substrategy DGO-1

Case 3: $ext(f_u)$ intersects $ext(f_r)$ in one point
Case 3.1: $(ext(f_u), ext(f_r)) \in \mathcal{AR}$
Go to the intersection point between $ext(f_u)$ and $ext(f_r)$
Case 3.2: $(ext(f_u), ext(f_r)) \in \mathcal{RB} \cup \mathcal{LA}$
Apply substrategy DGO-1
Case 3.3: $(ext(f_u), ext(f_r)) \in \mathcal{LB}$
Apply substrategy DGO-4

Case 4: $reg(f_u)$ and $reg(f_r)$ are disjoint
Case 4.1: $(ext(f_u), ext(f_r)) \in \mathcal{AA} \cup \mathcal{AR} \cup \mathcal{RR} \cup \mathcal{RA}$
Apply substrategy DGO-2
Case 4.2: $(ext(f_u), ext(f_r)) \in \mathcal{LR} \cup \mathcal{LA} \cup \mathcal{AB} \cup \mathcal{RB}$
Apply substrategy DGO-3
Case 4.3: $(ext(f_u), ext(f_r)) \in \mathcal{LB}$
Apply substrategy DGO-4
end while
if P is not completely visible
then Apply substrategy DGO-0
end if

LEMMA 4.2 *If q and q' are points on some tour SWR_s lying in $\mathbf{Q1}(s)$, then*

$$length(SWR_s) \geq 2\max(||s,q||_x, ||s,q'||_x) + 2\max(||s,q||_y, ||s,q'||_y).$$

PROOF: Since SWR_s passes through q and q' and we calculate length according to the L_1 metric, the smallest possible tour that passes through the two points is the rectangle with at least two of the points s, q, and q' on the perimeter. Since q and q' lie in $\mathbf{Q1}(s)$, the point s has to be the lower left corner of the rectangle and the length of the perimeter of the rectangle is then as stated. □

The structure of the following proofs are very similar to each other. In each case we will establish a point t that we can ensure is passed by SWR_s and that either lies on the boundary of $\mathbf{P}$ or can be viewed as to lie on the boundary of $\mathbf{P}$. We then consider the tour SWR_t (specified by strategy BGO) and compare how much longer the tour FR_s is. By Lemma 4.1 we know that $length(SWR_t) \leq length(SWR_s)$, hence the ratio

$$\frac{length(FR_s)}{length(SWR_t)}$$

is an upper bound on the competitive ratio of strategy DGO. One quirk of the proofs is that we require that there exists a tour SWR_t that contains the starting point s in its interior or on the boundary of SWR_t. This requirement will be removed in Lemma 4.8.

4.1 Strategy DGO-0

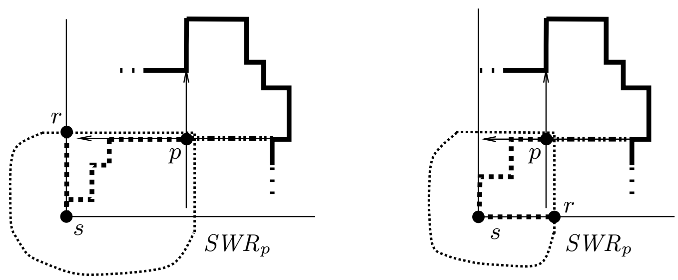
We start by presenting DGO-0.

Strategy DGO-0
1 Let p be the current robot position
2 **if** $||s,p||_y \leq ||s,p||_x$ **then**
2.1 $r :=$ PIGO(p, *east*, *cw*, $\mathbf{P}$)
else /* **if** $||s,p||_y > ||s,p||_x$ **then** */
2.2 $r :=$ PIGO(p, *east*, *aw*, $\mathbf{P}$)
endif
3 From r go back to the starting point s and halt
End DGO-0

LEMMA 4.3 *If the strategy applies substrategy* DGO-0*, then*

$$length(FR_s) \leq \frac{3}{2} length(SWR_s).$$

PROOF: Since the path $FR_s(s, p, orient)$ that the strategy has followed when it reaches point p is x-y-monotone, the point p is the currently topmost and rightmost point of the path. Hence, we can add a horizontal spike issuing from the boundary point immediately to the right of p, giving a new polygon $\mathbf{P}'$ having p on the boundary and furthermore with the same shortest watchman route through p as $\mathbf{P}$; see Figure 5. This means that performing strategy IGO(p, *east*, *orient*) in $\mathbf{P}$ yields the same result as performing BGO(p, *orient*) in $\mathbf{P}'$, p being a boundary point in $\mathbf{P}'$, *orient* being either *cw* or *aw*. The tour followed is therefore a shortest watchman route through the point p in both $\mathbf{P}'$ and $\mathbf{P}$.



$||s,p||_y \leq ||s,p||_x$ $||s,p||_y > ||s,p||_x$

(a) (b)

Figure 5: Illustrating the cases in the proof of Lemma 4.3.

Also the point p lies on a left extension with respect to s, by the way p is defined, and it is the closest point to s such that all of $\mathbf{Q1}(s)$ has been seen by the path $FR_s(s, p, orient) = SP(s, p)$. Hence, there is a route SWR_s that passes through p and by Lemma 4.1 $length(SWR_p) \leq length(SWR_s)$.

Assume now for the first case that DGO-0 performs Step 2.1, i.e., that when FR_s reaches the point p, then $||s,p||_y \leq ||s,p||_x$; see Figure 5(a). The result for the second case in which DGO-0 performs Step 2.2 follows by symmetry. The tour followed equals

$$FR_s \quad = \quad SP(s,p) \cup SWR_p(p,r,cw) \cup SP(r,s). \tag{1}$$

Since by our assumption, that s does not lie outside SWR_p, we can assume without loss of generality that the point r is the last intersection point of SWR_p with the vertical boundary axis of $\mathbf{Q1}(s)$; see Figure 5(a).

We prove this as follows. Let the point r be as defined in the strategy DGO-0 and let r' be the last intersection point of SWR_p with the vertical boundary axis of $\mathbf{Q1}(s)$. By the way the strategy PIGO is defined the subpath $SP(s,p) \cup SWR_p(p,r,cw)$ sees all of $\mathbf{P}$. Hence, the subpath $SWR_p(r,p,cw)$ equals $SP(r,p)$ with r either lying in $\mathbf{Q2}(s)$ or $\mathbf{Q3}(s)$. If $r \in \mathbf{Q3}(s)$ then $s \in SWR_p$, and $r' = s$. If $r \in \mathbf{Q2}(s)$ then we can choose r' so that $||r,s||_y = ||r',s||_y$, and $SP(r,s) = SP(r,r') \cup SP(r',s)$. In both cases $length(SWR_p(p,r,cw) \cup SP(r,s)) = length(SWR_p(p,r',cw) \cup SP(r',s))$. This proves that we can use the point r' instead of the point r in (1).

We have that

$$\begin{aligned}
length(FR_s) &= ||s,p|| + length(SWR_p) - ||r,p|| + ||r,s|| \\
&\leq length(SWR_s) + ||s,p|| + ||r,s|| - ||r,p|| \\
&= length(SWR_s) + ||s,p||_x + ||s,p||_y + ||r,s||_x + ||r,s||_y - ||r,p||_x - ||r,p||_y \\
&= length(SWR_s) + ||s,p||_y + ||r,s||_y - ||r,p||_y \quad \leq \quad length(SWR_s) + 2||s,p||_y \\
&\leq length(SWR_s) + ||s,p|| \quad \leq \quad \frac{3}{2} length(SWR_s),
\end{aligned}$$

since $length(SWR_s) \geq 2||s,p||$ by Lemma 4.2 and $||s,p||_y \leq ||s,p||_x$.

□

4.2 Strategy DGO-1

Next we present DGO-1.

Strategy DGO-1
1 **if** $ext(f_u) \in \mathcal{L}$ **then** Mirror **P** at the diagonal of $\mathbf{Q1}(s)$ **endif**
2 Go to v, the vertex immediately to the right of the current point
3 $q :=$ PBGO$(v, aw, \mathbf{Q1}(v))$
4 **if** $||s,q||_y \leq ||s,q||_x$ **then**
4.1 Go back to v
4.2 $r :=$ PBGO$(v, cw, \mathbf{P})$
else /* **if** $||s,q||_y > ||s,q||_x$ **then** */
4.3 $r :=$ PIGO$(q, north, aw, \mathbf{P})$
endif
5 Go back to the starting point s and halt
End DGO-1

Lemma 4.4 *If the strategy applies substrategy* DGO-1, *then*

$$length(FR_s) \leq \frac{3}{2} length(SWR_s).$$

Proof: Since the point v reached by path $FR_s(s, v, orient) = SP(s, v)$ is a boundary point, the result of performing strategy BGO$(v, orient)$, with $orient$ being either cw or aw, is a shortest watchman route through v; see Figure 6.

Also the point v lies on a shortest path between the horizontal boundary of $\mathbf{Q1}(s)$ and the extension $ext(f_u)$. Hence, there is a route SWR_s that passes through v and by Lemma 4.1 $length(SWR_v) \leq length(SWR_s)$.

Assume now for the first case that DGO-1 performs Steps 4.1 and 4.2, i.e., that when FR_s reaches the point q, then $||s,q||_y \leq ||s,q||_x$; see Figure 6(a). The tour followed equals

$$FR_s = SP(s,v) \cup SWR_v(v,q,aw) \cup SP(q,v) \cup SWR_v(v,r,cw) \cup SP(r,s)$$

and since by our assumption, that SWR_v contains s in its interior, we can assume without loss of generality that the point r is the last intersection point of SWR_v with the vertical boundary axis of $\mathbf{Q1}(s)$; see Figure 6(a). This can be proved in the same way as in Lemma 4.3. We have

$$\begin{aligned}
length(FR_s) &= ||s,q|| + length(SWR_v) - ||r,q|| + ||r,s|| \\
&\leq length(SWR_s) + ||s,q|| + ||r,s|| - ||r,q|| \\
&= length(SWR_s) + ||s,q||_x + ||s,q||_y + ||r,s||_x + ||r,s||_y - ||r,q||_x - ||r,q||_y \\
&= length(SWR_s) + ||s,q||_y + ||r,s||_y - ||r,q||_y \\
&= length(SWR_s) + 2||s,q||_y \quad \leq \quad \frac{3}{2} length(SWR_s),
\end{aligned}$$

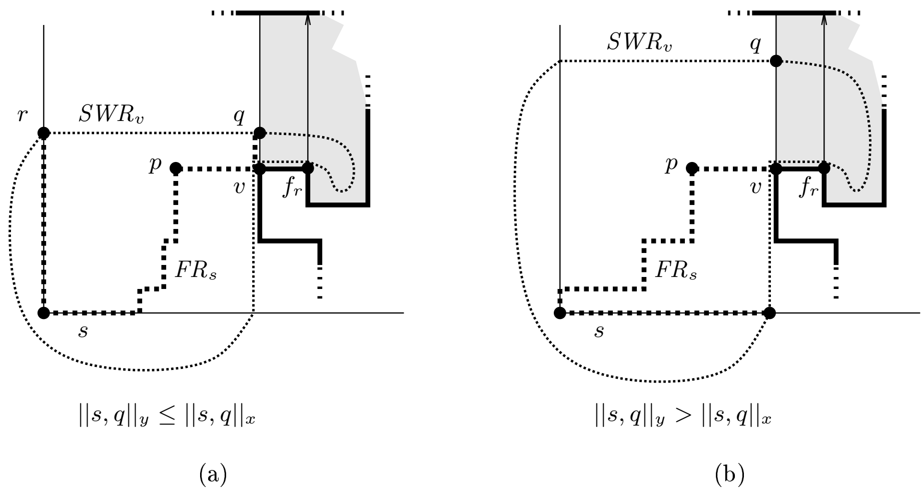


Figure 6: Illustrating the cases in the proof of Lemma 4.4.

since $length(SWR_s) \geq 2||s,q||$ by Lemma 4.2 and $||s,q||_y \leq ||s,q||_x$.

Assume for the second case that DGO-1 performs Step 4.3, i.e., once FR_s reaches the point q, then $||s,q||_y > ||s,q||_x$; see Figure 6(b). The tour followed equals

$$FR_s = SP(s,v) \cup SWR_v(v,r,aw) \cup SP(r,s),$$

with r being the last intersection point of SWR_v with the horizontal boundary axis of $\mathbf{Q1}(s)$; see Figure 6(b). In the same way as before we can now obtain

$$length(FR_s) \quad \leq \quad length(SWR_s) + 2||s,q||_x \quad \leq \quad \frac{3}{2} length(SWR_s).$$

This concludes the proof. □

4.3 Strategy DGO-2

We continue the analysis by first showing the substrategy DGO-2 and then proving its competitive ratio.

Strategy DGO-2

1 Let e be the leftmost vertical edge in $\mathbf{Q1}(s)$ on the boundary part separating $ext(f_u)$ and $ext(f_r)$. Similarly let e' be the bottommost horizontal edge in $\mathbf{Q1}(s)$ on the boundary part separating $ext(f_u)$ and $ext(f_r)$

2 Let v be the topmost vertex on e and let v' be the rightmost vertex on e'

3 **if** $||s,v||_y < ||s,v'||_x$ **then**

3.1 Mirror $\mathbf{P}$ at the diagonal of $\mathbf{Q1}(s)$

/* v and v' are now swapped */

endif

4 Go to v

5 $q :=$ PBGO$(v, aw, \mathbf{Q1}(v))$

6 **if** $||s,v|| \leq ||s,q||_y$ **then**

6.1 $r :=$ PIGO$(q, north, aw, \mathbf{P})$

else /* **if** $||s,v|| > ||s,q||_y$ **then** */

6.2 Go back to v

6.3 $r :=$ PBGO$(v, cw, \mathbf{P})$

endif

7 Go back to s and halt

End DGO-2

LEMMA 4.5 *If the strategy applies substrategy* DGO-2, *then*

$$length(FR_s) \leq \frac{5}{3} length(SWR_s).$$

PROOF: Before we begin the actual proof we have to ensure that the points v and v' defined in substrategy DGO-2 are such that in all cases it holds that $length(SWR_v) \leq length(SWR_s)$ and $length(SWR_{v'}) \leq length(SWR_s)$. In cases $\mathcal{AA}$ and $\mathcal{AR}$, the point v lies on some tour SWR_s since v lies on a shortest path from s to $ext(f_u)$ and on a shortest path from $ext(f_r)$ to $ext(f_u)$. Hence, for these two cases it follows by Lemma 4.1 that $length(SWR_v) \leq length(SWR_s)$; see Figures 7(a) and (b). Furthermore, since v' lies on some tour SWR_v since v' lies on a shortest path from $ext(f_r)$ to $ext(f_u)$, we have by the same token that $length(SWR_{v'}) \leq length(SWR_v)$, thus proving the second inequality.

The case $\mathcal{RR}$ follows by a completely symmetric argument if we exchange the points v and v' for each other; see Figure 7(c).

The final case $\mathcal{RA}$ is a little more involved since neither v nor v' can be ensured to lie on a tour SWR_s. To prove the inequalities consider an intersection point u between some tour SWR_s and $ext(f_r)$. The point u must exist otherwise SWR_s violates the visibility property of a watchman route. Now, by Lemma 4.1, we have that $length(SWR_u) \leq length(SWR_s)$. Since v and v' lie on some shortest path from u to $ext(f_u)$ we additionally have that there is some tour SWR_u passing through both these points. Applying Lemma 4.1 again gives us the two inequalities $length(SWR_v) \leq length(SWR_u)$ and $length(SWR_{v'}) \leq length(SWR_u)$, thus proving our claim also in this case; see Figure 7(d).

Assume for the first case that DGO-2 performs Step 6.1, i.e., that $||s,v||_y \leq ||s,v'||_x$ and that when FR_s reaches the point q, then $||s,v|| \leq ||s,q||_y$; see Figure 8(a). The tour followed equals

$$FR_s = SP(s,v) \cup SWR_v(v,r,aw) \cup SP(r,s).$$

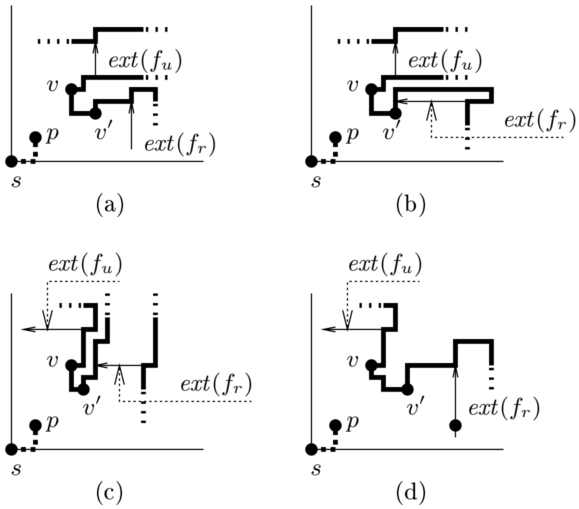


Figure 7: Illustrating the structural cases handled by substrategy DGO-2.

Hence, we have that the length of FR_s is bounded by

$$\begin{aligned}
\mathit{length}(FR_s) &= ||s,v|| + \mathit{length}(SWR_v(v,r,aw)) + ||r,s|| \\
&\leq \mathit{length}(SWR_v) + 2||s,v|| \quad \leq \quad \mathit{length}(SWR_s) + \frac{4}{3}||s,v|| + \frac{2}{3}||s,v|| \\
&\leq \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{2}{3}||s,v||_x + \frac{2}{3}||s,v||_y \\
&\leq \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{2}{3}||s,v'||_x + \frac{2}{3}||s,v'||_x \\
&= \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{4}{3}||s,v'||_x \\
&\leq \mathit{length}(SWR_s) + \frac{2}{3}\mathit{length}(SWR_s) \quad = \quad \frac{5}{3}\mathit{length}(SWR_s),
\end{aligned}$$

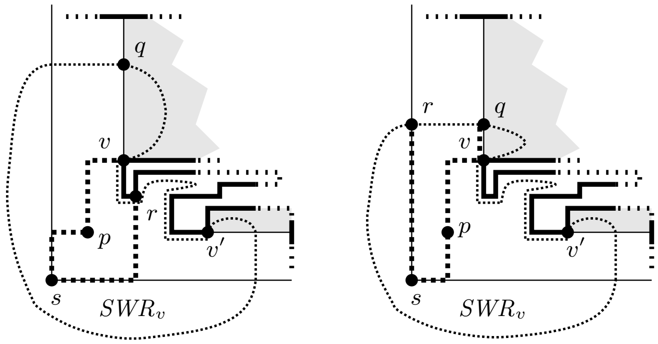
since $\mathit{length}(SWR_s) \geq 2||s,q||_y + 2||s,v'||_x$ by Lemma 4.2.

For the second case assume that DGO-2 performs Steps 6.2 and 6.3, i.e., that $||s,v||_y \leq ||s,v'||_x$ and that when FR_s reaches the point q, then $||s,v|| > ||s,q||_y$; see Figure 8(b). The tour followed equals

$$FR_s = SP(s,v) \cup SWR_v(v,q,aw) \cup SP(q,v) \cup SWR_v(v,r,cw) \cup SP(r,s).$$

Without loss of generality we can assume that r is the last intersection point between SWR_v and the vertical boundary axis of $\mathbf{Q1}(s)$. The length of FR_s is bounded by

$$\begin{aligned}
\mathit{length}(FR_s) &= ||s,q|| + \mathit{length}(SWR_v(v,q,aw)) + \mathit{length}(SWR_v(v,r,cw)) + ||r,s|| \\
&= ||s,q|| + \mathit{length}(SWR_v) - ||r,q|| + ||r,s|| \\
&\leq \mathit{length}(SWR_s) + ||s,q||_x + ||s,q||_y + ||r,s||_x + ||r,s||_y - ||r,q||_x - ||r,q||_y \\
&= \mathit{length}(SWR_s) + ||s,q||_y + ||r,s||_y \quad \leq \quad \mathit{length}(SWR_s) + 2||s,q||_y
\end{aligned}$$



$||s,v||_y \leq ||s,v'||_x$
$||s,v|| \leq ||s,q||_y$

(a)

$||s,v||_y \leq ||s,v'||_x$
$||s,v|| > ||s,q||_y$

(b)

Figure 8: Illustrating the cases in the proof of Lemma 4.5.

$$
\begin{aligned}
&= \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{2}{3}||s,q||_y \quad < \quad \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{2}{3}||s,v|| \\
&= \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{2}{3}||s,v||_x + \frac{2}{3}||s,v||_y \\
&\leq \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{2}{3}||s,v'||_x + \frac{2}{3}||s,v'||_x \\
&= \mathit{length}(SWR_s) + \frac{4}{3}||s,q||_y + \frac{4}{3}||s,v'||_x \\
&\leq \mathit{length}(SWR_s) + \frac{2}{3}\mathit{length}(SWR_s) \quad = \quad \frac{5}{3}\mathit{length}(SWR_s),
\end{aligned}
$$

since $\mathit{length}(SWR_s) \geq 2||s,q||_y + 2||s,v'||_x$ by Lemma 4.2.

This concludes the proof. □

4.4 Strategy DGO-3

We proceed with substrategy DGO-3.

Strategy DGO-3

1 **if** $(ext(f_u), ext(f_r)) \in \mathcal{LR} \cup \mathcal{LA}$ **then** Mirror **P** at the diagonal of $\mathbf{Q1}(s)$ **endif**

2 Let e be the leftmost vertical edge in $\mathbf{Q1}(s)$ on the boundary part separating $ext(f_u)$ and $ext(f_r)$ and let v be the topmost vertex on e. Let v' be the vertex immediately to the right of the current point

3 **if** $||s, v'||_x \leq ||s, v||_x$ **then**

3.1 Go to v'

3.2 $q :=$ PBGO$(v', aw, \mathbf{Q1}(v'))$

3.3 **if** $||s, q||_y \leq ||s, q||_x$ **then**

3.3.1 Go back to v'

3.3.2 $r :=$ PBGO$(v', cw, \mathbf{P})$

else /* **if** $||s, q||_y > ||s, q||_x$ **then** */

3.3.3 $r :=$ PIGO$(q, north, aw, \mathbf{P})$

endif

else /* **if** $||s, v'||_x > ||s, v||_x$ **then** */

3.4 **if** $||s, v||_y \leq ||s, v'||_x$ **then**

3.4.1 Go to v

3.4.2 $q :=$ PBGO$(v, aw, \mathbf{Q1}(v))$

3.4.3 **if** $||s, v|| \leq ||s, q||_y$ **then**

3.4.3.1 $r :=$ PIGO$(q, north, aw, \mathbf{P})$

else /* **if** $||s, v|| > ||s, q||_y$ **then** */

3.4.3.2 Go back to v

3.4.3.3 $r :=$ PBGO$(v, cw, \mathbf{P})$

endif

else /* **if** $||s, v||_y > ||s, v'||_x$ **then** */

3.5 Go to v'

3.6 $r :=$ PBGO$(v', aw, \mathbf{P})$

endif

endif

4 Go back to s and halt

End DGO-3

LEMMA 4.6 *If the strategy applies substrategy* DGO-3*, then*

$$length(FR_s) \leq \frac{5}{3} length(SWR_s).$$

PROOF: Since the point v' reached by path $FR_s(s, v', orient) = SP(s, v')$ is a boundary point the result of performing strategy BGO$(v', orient)$, with *orient* being either *cw* or *aw*, is a shortest watchman route through v'; see Figure 9.

Also, the point v' lies on a shortest path between the horizontal boundary of $\mathbf{Q1}(s)$ and the extension $ext(f_r)$. Hence, there is a route SWR_s that passes through v' and by Lemma 4.1 $length(SWR_{v'}) \leq length(SWR_s)$. The point v lies on a shortest path from $ext(f_r)$ to $ext(f_u)$ and therefore we have that there is a route SWR_s that passes through v and $length(SWR_v) \leq length(SWR_s)$.

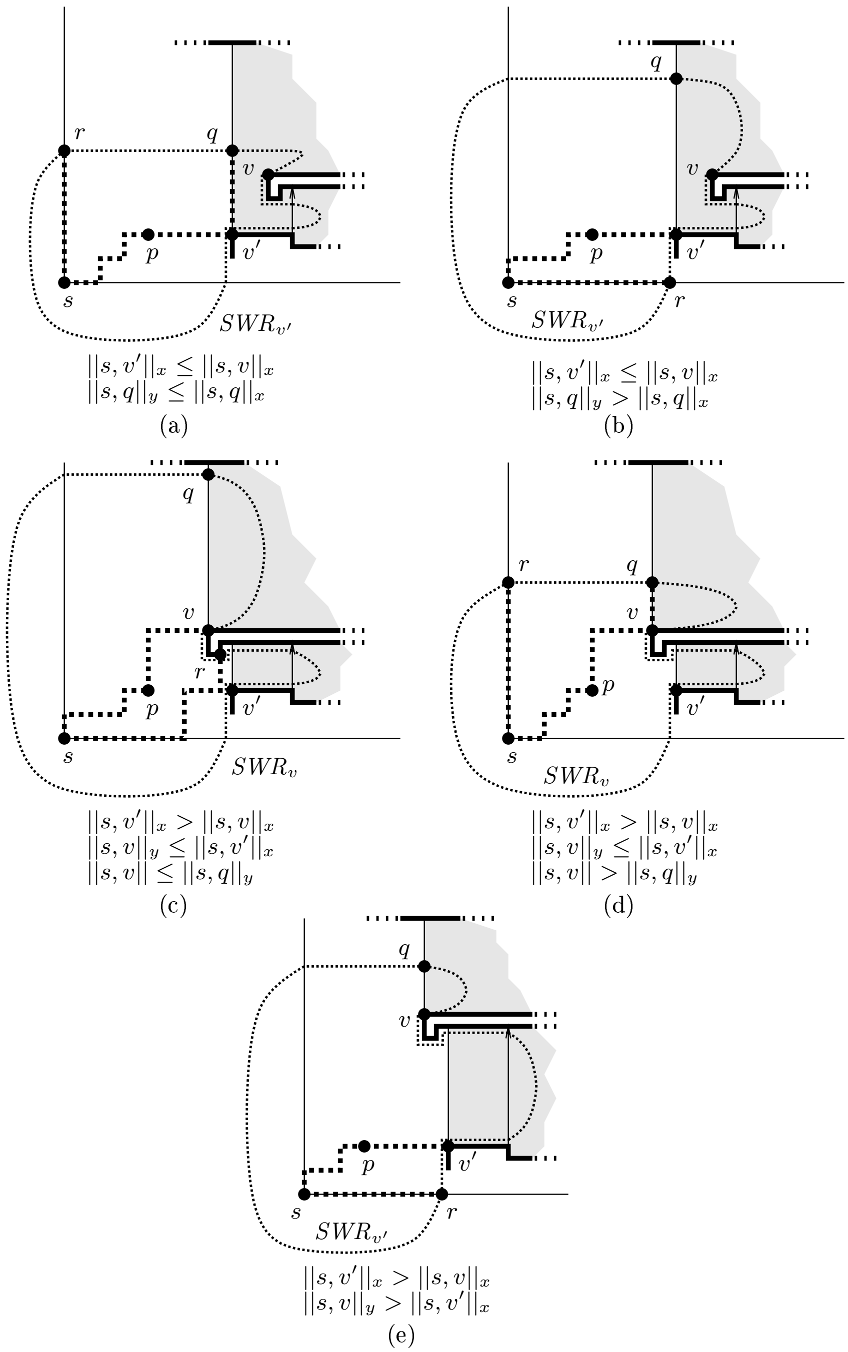


Figure 9: Illustrating the cases in the proof of Lemma 4.6.

Assume for the first case that DGO-3 performs Steps 3.3.1 and 3.3.2, i.e., that $||s, v'||_x \leq ||s, v||_x$ and $||s, q||_y \leq ||s, q||_x$; see Figure 9(a). The tour followed equals

$$FR_s = SP(s, v') \cup SWR_{v'}(v', q, aw) \cup SP(q, v') \cup SWR_{v'}(v', r, cw) \cup SP(r, s),$$

with r being the last intersection point of $SWR_{v'}$ with the vertical boundary axis of $\mathbf{Q1}(s)$. In the case that the strategy performs Step 3.3.3 i.e., that $||s, v'||_x \leq ||s, v||_x$ and $||s, q||_y > ||s, q||_x$; see Figure 9(b). The tour followed equals

$$FR_s = SP(s, v') \cup SWR_{v'}(v', r, aw) \cup SP(r, s),$$

with r being the last intersection point of $SWR_{v'}$ with the horizontal boundary axis of $\mathbf{Q1}(s)$. In the same way as in the proof of Lemma 4.4 we can obtain a competitive ratio of 3/2 in this case.

Assume for the second case that DGO-3 performs Step 3.4.3.1, i.e., that $||s, v'||_x > ||s, v||_x$, $||s, v||_y \leq ||s, v'||_x$ and $||s, v|| \leq ||s, q||_y$; see Figure 9(c). The tour followed equals

$$FR_s = SP(s, v) \cup SWR_v(v, r, aw) \cup SP(r, s),$$

with r being the last intersection point of SWR_v with the horizontal boundary axis of $\mathbf{Q1}(s)$. In the case that the strategy performs Steps 3.4.3.2 and 3.4.3.3, i.e., that $||s, v'||_x > ||s, v||_x$, $||s, v||_y \leq ||s, v'||_x$ and $||s, v|| > ||s, q||_y$; see Figure 9(d). The tour followed equals

$$FR_s = SP(s, v) \cup SWR_v(v, q, aw) \cup SP(q, v) \cup SWR_v(v, r, cw) \cup SP(r, s)$$

with r being the last intersection point of SWR_v with the vertical boundary axis of $\mathbf{Q1}(s)$. In the same way as in the proof of Lemma 4.5 we can obtain a competitive ratio of 5/3 in this case.

Assume for the third case that DGO-3 performs Steps 3.5 and 3.6, i.e., that $||s, v'||_x > ||s, v||_x$ and $||s, v||_y > ||s, v'||_x$; see Figure 9(e). The tour followed equals

$$FR_s = SP(s, v') \cup SWR_{v'}(v', r, aw) \cup SP(r, s),$$

$$\begin{aligned}
\mathit{length}(FR_s) &= ||s, v'|| + \mathit{length}(SWR_{v'}(v', r, aw)) + ||r, s|| \\
&\leq ||s, v'|| + \mathit{length}(SWR_{v'}) - ||r, v'|| + ||r, s|| \quad \leq \quad \mathit{length}(SWR_s) + 2||s, v'||_x \\
&< \mathit{length}(SWR_s) + ||s, v'||_x + ||s, v||_y \quad \leq \quad \frac{3}{2}\mathit{length}(SWR_s),
\end{aligned}$$

since $\mathit{length}(SWR_s) \geq 2||s, v'||_x + 2||s, v||_y$ by Lemma 4.2. From our assumption, that $SWR_{v'}$ contains s in its interior, we can assume without loss of generality that the point r is the last intersection point of $SWR_{v'}$ with the vertical boundary axis of $\mathbf{Q1}(s)$. This can be proved in the same way as in Lemma 4.3.

This concludes the proof. □

4.5 Strategy DGO-4

We conclude with substrategy DGO-4.

Strategy DGO-4

1 Let v be the vertex immediately above p and let v' be the vertex immediately to the right of p.

2 **if** $||s,v||_y \leq ||s,v'||_x$ **then**

2.1 Go to v

2.2 $r :=$ PBGO$(v, cw, \mathbf{P})$

else /* **if** $||s,v||_y > ||s,v'||_x$ **then** */

2.3 Go to v'

2.4 $r :=$ PBGO$(v', aw, \mathbf{P})$

endif

3 Go back to s and halt

End DGO-4

LEMMA 4.7 *If the strategy applies substrategy* DGO-4*, then*

$$\mathit{length}(FR_s) \leq \frac{3}{2}\mathit{length}(SWR_s).$$

PROOF: Consider the point v. It lies on the shortest path between the horizontal boundary of $\mathbf{Q1}(s)$ and the extension $\mathit{ext}(f_u)$. Hence, there is a route SWR_s that passes through v and by Lemma 4.1 $\mathit{length}(SWR_v) \leq \mathit{length}(SWR_s)$. By a completely symmetric argument it also follows that there is a route SWR_s that passes through v', and hence, $\mathit{length}(SWR_{v'}) \leq \mathit{length}(SWR_s)$.

Assume first that DGO-4 performs Steps 2.1 and 2.2, i.e., $||s,v||_y \leq ||s,v'||_x$; see Figure 10(a). The tour followed equals

$$FR_s = SP(s,v) \cup SWR_v(v,r,cw) \cup SP(r,s).$$

Without loss of generality we can assume that r is the last intersection point between SWR_v and the vertical boundary axis of $\mathbf{Q1}(s)$. This can be proved in the same way as in Lemma 4.3. The length of FR_s is bounded by

$$\begin{aligned} \mathit{length}(FR_s) &= ||s,v|| + \mathit{length}(SWR_v(v,r,cw)) + ||r,s|| \\ &\leq ||s,v|| + \mathit{length}(SWR_v) - ||r,v|| + ||r,s|| \\ &\leq \mathit{length}(SWR_s) - ||r,v|| + ||s,v|| + ||r,s|| \\ &\leq \mathit{length}(SWR_s) + 2||s,v||_y \\ &\leq \mathit{length}(SWR_s) + ||s,v||_y + ||s,v'||_x \\ &\leq \frac{3}{2}\mathit{length}(SWR_s), \end{aligned}$$

since $\mathit{length}(SWR_s) \leq 2||s,v||_y + 2||s,v'||_x$ by Lemma 4.2.

If DGO-4 performs Steps 2.3 and 2.4, i.e., $||s,v||_y > ||s,v'||_x$; see Figure 10(b); the tour followed equals

$$FR_s = SP(s,v') \cup SWR_{v'}(v',r,aw) \cup SP(r,s)$$

and similarly we can assume that r is the last intersection point between $SWR_{v'}$ and the horizontal boundary axis of $\mathbf{Q1}(s)$ as proved in Lemma 4.3. In the same way as before we obtain that

$$\mathit{length}(FR_s) \leq \mathit{length}(SWR_s) + 2||s,v'||_x \leq \frac{3}{2}\mathit{length}(SWR_s).$$

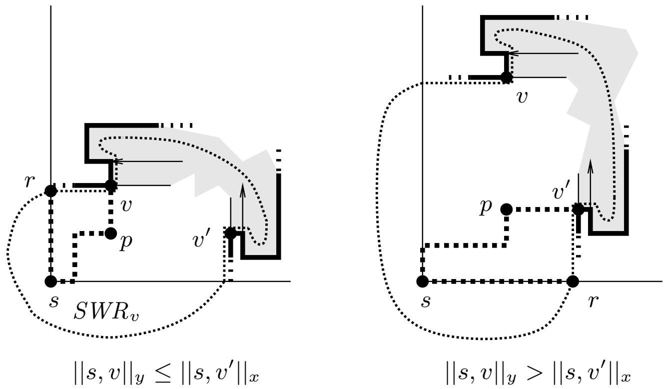


Figure 10: Illustrating the cases in the proof of Lemma 4.7.

This concludes the proof. □

In each of the proofs of Lemmas 4.3–4.7 we established a key point, from now on denoted t, in $\mathbf{Q1}(s)$ that is passed by some tour SWR_s. If SWR_t contains the starting point s in the interior or on the boundary then the results of Lemmas 4.3–4.7 hold as we have shown. Now consider the case when s lies outside SWR_t. We need to show that the results of the lemmas still hold in this case.

Lemma 4.8 *Using the notation above, the results of Lemmas 4.3–4.7 still hold if s is not contained in SWR_t.*

Proof: Of all the possible tours SWR_t consider one that has a point s' visible from s and as close to s as possible. We will show that

$$\mathit{length}(SWR_s) \geq 2||s,s'|| + \mathit{length}(SWR_{s'})$$

and that

$$\mathit{length}(FR_s) \leq 2||s,s'|| + \mathit{length}(FR_{s'}).$$

Thus, we have that

$$\begin{aligned} \mathit{length}(FR_s) &\leq 2||s,s'|| + \mathit{length}(FR_{s'}) \leq 2||s,s'|| + c \cdot \mathit{length}(SWR_{s'}) \\ &\leq 2c||s,s'|| + c \cdot \mathit{length}(SWR_{s'}) \leq c \cdot \mathit{length}(SWR_s) \end{aligned}$$

for any $c \geq 1$ such that $\mathit{length}(FR_{s'}) \leq c \cdot \mathit{length}(SWR_{s'})$. Hence, we can apply $c = 3/2$ or $c = 5/3$ on each of the cases DGO-0 to DGO-4 appropriately since the point s' is by definition contained on the boundary of SWR_t.

To prove the two inequalities, we first assume that SWR_t is contained in $\mathbf{Q1}(s)$ and possibly also $\mathbf{Q4}(s)$. The only other possibility is that SWR_t is contained in $\mathbf{Q1}(s)$ and $\mathbf{Q2}(s)$ which is

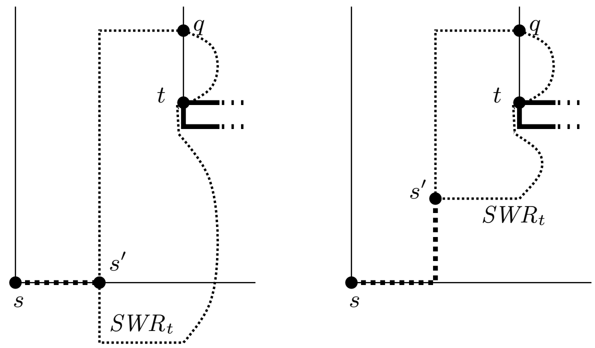


Figure 11: Illustrating the proof of Lemma 4.8.

symmetrical by a mirroring operation of the polygon **P** at the diagonal of $\mathbf{Q1}(s)$ and therefore can be handled in the same way.

Next, note that the point s' lies on an x-y-monotone path from s to t. This is because s' cannot lie above the y-coordinate of t and it cannot lie to the right of the x-coordinate of t. If s' did, then it would be possible to move it closer to s contradicting our initial assumption. (Indeed if SWR_t lies in both $\mathbf{Q1}(s)$ and $\mathbf{Q4}(s)$, then s' lies on the horizontal boundary of $\mathbf{Q1}(s)$.)

Consider a tour SWR_s. Since SWR_t has no points in $\mathbf{Q2}(s')$ or $\mathbf{Q3}(s')$ it follows that these quadrants cannot contain any vertical left extensions with respect to s'. Hence, it is possible to move any vertical edge of SWR_s (partially) visible from s and to the left of s' towards the right so that the new tour constructed is a watchman route, passes through s', has the same length as SWR_s, and has no vertical edge in $\mathbf{Q2}(s')$ or $\mathbf{Q3}(s')$; see Figures 11(a) and (b). The tour constructed in this way consists of a watchman route passing through s' and an x-y-monotone path connecting s and s'. Exchanging the watchman route passing through s' for a tour $SWR_{s'}$ possibly shortens the total length of the tour and this proves the first inequality.

To prove the second inequality we note that FR_s consists of an x-y-monotone path from s to t followed by a part of SWR_t and a path back to s. Hence we can (for this proof only) assume that the initial path from s to t passes through s'. Furthermore we note that whichever substrategy is applied for FR_s the same substrategy will be applied for $FR_{s'}$. However within a substrategy not necessarily the same steps will be performed for FR_s and $FR_{s'}$. We define this difference formally as follows.

Consider the tour FR_p for any point p. If DGO on p applies a substrategy that does one of the following:

1. its first application of PIGO or PBGO is with anti-clockwise orientation and the (possible) subsequent application of PIGO or PBGO is also with anti-clockwise orientation,

2. its first application of PIGO or PBGO is with clockwise orientation and the subsequent application of PIGO or PBGO is with anti-clockwise orientation,

we call the tour *forward oriented* otherwise the tour is *backward oriented.*

That is, independently of which substrategy that is applied we continue the path from the current point to t, apply one of PIGO or PBGO until we reach a point q where a decision is made depending on whether $||s,q||_y \leq ||s,q||_x$ or $||s,q||_y > ||s,q||_x$. This decision is precisely the one that determines whether the tour will be forward oriented or backward oriented.

If both FR_s and $FR_{s'}$ are forward oriented or backward oriented, then the second inequality follows directly. Hence, we have a problem if either FR_s is forward oriented whereas $FR_{s'}$ is backward oriented or the reverse case occurs. We do a case analysis to prove the second inequality.

Assume first that FR_s is backward oriented and $FR_{s'}$ is forward oriented. Let o denote the orientation that the first application of PIGO or PBGO in DGO on s performs in a substrategy. Let $\bar{o}$ be the other orientation, i.e., if $o = aw$, then $\bar{o} = cw$ and conversely.

We have that

$$\begin{aligned} \mathit{length}(FR_s) &= ||s,s'|| + \mathit{length}(FR_s(s',q,o)) + ||q,t|| + ||t,r|| + \\ &\quad + \mathit{length}(FR_s(r,s',\bar{o})) + ||s',s|| \\ &= \mathit{length}(FR_{s'}(s',q,o)) + ||q,s'|| + \mathit{length}(FR_{s'}(s',r,o)) + ||r,s'|| + \\ &\quad + 2||s,s'|| + ||t,r|| + ||q,t|| - ||q,s'|| - ||r,s'|| \\ &= \mathit{length}(FR_{s'}) + 2||s,s'|| + ||t,r|| + ||q,t|| - ||q,s'|| - ||r,s'|| \\ &\leq \mathit{length}(FR_{s'}) + 2||s,s'||, \end{aligned}$$

since $\mathit{length}(FR_s(r,s',\bar{o})) = \mathit{length}(FR_{s'}(s',r,o))$, $\mathit{length}(FR_s(s',q,o)) = \mathit{length}(FR_{s'}(s',q,o))$, and

$$\begin{aligned} ||t,r|| + ||q,t|| - ||q,s'|| - ||r,s'|| &= \\ ||t,r||_x + ||t,r||_y + ||q,t||_x + ||q,t||_y - ||q,s'||_x - ||q,s'||_y - ||r,s'||_x - ||r,s'||_y &= \\ ||t,r||_y + ||q,t||_y - ||q,s'||_x - ||q,s'||_y - ||r,s'||_x - ||r,s'||_y &= \\ ||r,q||_y - ||q,s'||_x - ||q,s'||_y - ||r,s'||_x - ||r,s'||_y &\leq \\ -||q,s'||_x - ||r,s'||_x - ||r,s'||_y &\leq 0, \end{aligned}$$

if $o = aw$; see Figure 12(a). The penultimate inequality follows since $||s',q||_y \geq ||r,q||_y$. If $o = cw$, then a similar calculation can easily be made.

Assume now that FR_s is forward oriented and $FR_{s'}$ is backward oriented. If SWR_t intersects both $\mathbf{Q1}(s)$ and $\mathbf{Q4}(s)$ (this means that s' lies on the horizontal boundary of $\mathbf{Q1}(s)$), we argue as follows. Since $||s,q||_y = ||s',q||_y$ in this case, this means that if $||s,q||_x < ||s,q||_y$, then $||s',q||_x < ||s',q||_y$, and hence, it is not possible for FR_s to be forward oriented and $FR_{s'}$ to be backward oriented. Therefore SWR_t must be completely contained in $\mathbf{Q1}(s)$.

We can furthermore assume that s' is selected so that SWR_t lies completely in $\mathbf{Q1}(s')$. We have, if the substrategy used is not DGO-4, that

$$\begin{aligned} \mathit{length}(FR_s) &= ||s,s'|| + \mathit{length}(FR_s(s',q,o)) + ||q,t|| + ||t,r'|| + \\ &\quad + \mathit{length}(FR_s(r',r,o)) + ||r,s'|| + ||s',s|| \\ &= \mathit{length}(FR_{s'}(s',q,o)) + ||q,t|| + ||t,r|| + \mathit{length}(FR_{s'}(r,r',\bar{o})) + ||r',s'|| + \\ &\quad + 2||s',s|| + ||r,s'|| + ||t,r'|| - ||r',s'|| - ||t,r|| \\ &= \mathit{length}(FR_{s'}) + 2||s,s'|| + ||r,s'|| + ||t,r'|| - ||r',s'|| - ||t,r|| \end{aligned}$$

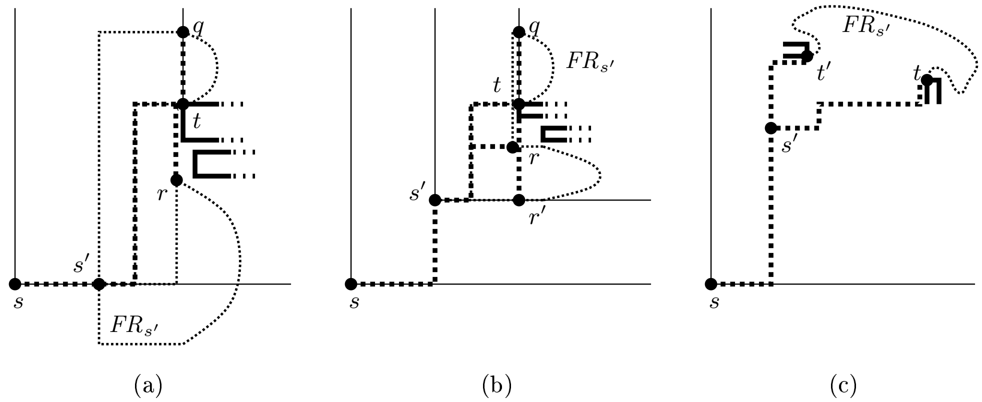


Figure 12: Illustrating the proof of Lemma 4.8.

$$\leq \quad \mathit{length}(FR_{s'}) + 2||s, s'||,$$

since $\mathit{length}(FR_{s'}(r, r', \bar{o})) = \mathit{length}(FR_s(r', r, o))$, $\mathit{length}(FR_s(s', q, o)) = \mathit{length}(FR_{s'}(s', q, o))$, and $||r, s'|| + ||t, r'|| - ||r', s'|| - ||t, r|| \leq 0$ by a similar calculation as above; see Figure 12(b) for an example of this case when $o = aw$. The key insight is that once the point q is reached, then all of $\mathbf{Q1}(s)$ that also lies in $\mathbf{Q1}(t)$, $\mathbf{Q2}(t)$, and $\mathbf{Q3}(t)$ has already been seen. Hence it only remains to explore the part in $\mathbf{Q4}(t)$.

Finally, if the substrategy used is DGO-4, then we have two different key points, the point t reached by FR_s and the point t' reached by $FR_{s'}$, giving

$$\begin{aligned}
\mathit{length}(FR_s) &= ||s, s'|| + ||s', t|| + \mathit{length}(FR_s(t, t', aw)) + ||t', s'|| + ||s', s|| \\
&= ||s', t'|| + \mathit{length}(FR_{s'}(t', t, cw)) + ||t, s'|| + 2||s', s|| + \\
&\quad + ||s', t|| + ||t', s'|| - ||s', t'|| - ||t, s'|| \\
&= \mathit{length}(FR_{s'}) + 2||s', s|| + ||s', t|| + ||t', s'|| - ||s', t'|| - ||t, s'|| \\
&= \mathit{length}(FR_{s'}) + 2||s, s'||,
\end{aligned}$$

since $\mathit{length}(FR_s(t, t', aw)) = \mathit{length}(FR_{s'}(t', t, cw))$ and $||s', t|| + ||t', s'|| - ||s', t'|| - ||t, s'|| = 0$; see Figure 12(c).

This concludes the proof. □

Next, we prove the correctness of our strategy.

LEMMA 4.9 *If the strategy* DGO *diverts from an x-y-monotone path then one of the substrategies* DGO-0 *to* DGO-4 *is performed.*

PROOF: The objective of this proof is to ensure that all cases are taken care of. We enumerate all the possible cases and show that unless one of the substrategies DGO-0 to DGO-4 is entered then strategy will continue along an x-y-monotone path inside $\mathbf{P}$.

We begin by assuming that the current point of the robot is a point p in $\mathbf{Q1}(s)$. Furthermore, we assume that the path from s to p is x-y-monotone and that all of $\mathbf{Q1}(s)$ to the left and below

p has been seen so far. All these assumptions are true for the starting point, i.e., when $p = s$. Let $ext(f_u)$ and $ext(f_r)$ be the upper and right frontier established from the point p having their issuing edges inside $\mathbf{Q1}(s)$. This gives rise to three possible cases.

The first case is that neither $ext(f_u)$ nor $ext(f_r)$ exist. This means that the path from s to p sees all of $\mathbf{Q1}(s)$ and substrategy DGO-0 guarantees a competitive ratio of 3/2 in this case; see Lemma 4.3.

The second case is that the extensions $ext(f_u)$ and $ext(f_r)$ exist but $ext(f_u) = ext(f_r)$, i.e., in essence we only have one extension. If the extension is in $\mathcal{B}$ or $\mathcal{L}$, then substrategy DGO-1 guarantees a competitive ratio of 3/2 in this case; see Lemma 4.4. On the other hand, if the extension is in $\mathcal{A}$ or $\mathcal{R}$, then the robot moves to the closest point on the extension. This can be done with an x-y-monotone path moving upwards and to the right from the current point thus showing that the path from s to the new current point is x-y-monotone. Furthermore, everything in $\mathbf{Q1}(s)$ to the left and below the new current point has been seen.

The third case is that the extensions $ext(f_u)$ and $ext(f_r)$ exist and are different. This case has a number of subcases. If $ext(f_u)$ is in $\mathcal{B}$, then $ext(f_r)$ must also lie in $\mathcal{B}$, and hence, one extension dominates the other. Since $ext(f_u)$ lies in $\mathcal{B}$ is symmetric with respect to a mirroring operation of $\mathbf{P}$ along the diagonal of $\mathbf{Q1}(s)$ to $ext(f_r)$ lies in $\mathcal{L}$, we have by the same argument that one extension dominates the other in this case. So, if we assume that the two extensions do not dominate each other and that they do not intersect, we have that the pair of extensions can be in one of the following nine cases:

$$(ext(f_u), ext(f_r)) \in \mathcal{AA} \cup \mathcal{AB} \cup \mathcal{AR} \cup \mathcal{LA} \cup \mathcal{LB} \cup \mathcal{LR} \cup \mathcal{RA} \cup \mathcal{RB} \cup \mathcal{RR}.$$

If $(ext(f_u), ext(f_r)) \in \mathcal{AA} \cup \mathcal{AR} \cup \mathcal{RR} \cup \mathcal{RA}$, then substrategy DGO-2 guarantees a competitive ratio of 5/3 in this case; see Lemma 4.5.

If $(ext(f_u), ext(f_r)) \in \mathcal{LA} \cup \mathcal{LR} \cup \mathcal{RB} \cup \mathcal{AR}$, then substrategy DGO-3 guarantees a competitive ratio of 5/3 in this case; see Lemma 4.6.

If $(ext(f_u), ext(f_r)) \in \mathcal{LB}$, then substrategy DGO-4 guarantees a competitive ratio of 3/2 in this case; see Lemma 4.7.

Next, we assume that the two extensions intersect, hence they are orthogonal. The pair of extensions can be in one of the following five cases:

$$(ext(f_u), ext(f_r)) \in \mathcal{AR} \cup \mathcal{LA} \cup \mathcal{LB} \cup \mathcal{RA} \cup \mathcal{RB}.$$

The case $(ext(f_u), ext(f_r)) \in \mathcal{RA}$ cannot occur with the two extensions intersecting leaving us with the four remaining cases.

If $(ext(f_u), ext(f_r)) \in \mathcal{LA} \cup \mathcal{RB}$, then substrategy DGO-1 guarantees a competitive ratio of 3/2 in this case; see Lemma 4.6.

If $(ext(f_u), ext(f_r)) \in \mathcal{LB}$, then substrategy DGO-4 guarantees a competitive ratio of 3/2 in this case; see Lemma 4.7.

If $(ext(f_u), ext(f_r)) \in \mathcal{AR}$, then by our inductive assumption we can move the current point with an x-y-monotone path to the intersection point of the extensions. Everything to the left and below the intersection point is seen by an x-y-monotone path from s to the intersection point.

Finally, if one extension dominates the other, then we simply let the robot move to the closest point on the closest extension. This can be achieved by an x-y-monotone path from the current point. Also, we maintain the property that everything to the left and below the intersection point has been seen so far.

This concludes the case analysis and the proof. □

Lemmas 4.3–4.9 lead us to establish the total competitive ratio of DGO. We have the theorem.

THEOREM 1
$$length(FR_s) \leq \frac{5}{3} length(SWR_s).$$

PROOF: It only remains to prove the correctness of our strategy, i.e., that once the strategy terminates the complete polygon has been explored. But this follows from Lemma 2.1 since the strategy ensures that FR_s has at least one point to the right of every extension. □

5 Conclusions

We have proved a 5/3-competitive deterministic strategy called DGO for exploring a rectilinear polygon in the L_1 metric. We anticipate that with a similar method the competitive ratio should be possible to be improved to 3/2 although the details are yet to be ironed out. This new strategy makes extensive use of the strategy DGO and its substrategies DGO-0 to DGO-4.

Closing the gap to the known lower bound of 5/4 is still an open problem. We hope that our method will give new insight so that the gap can be narrowed further or even closed altogether.

References

[1] MARGRIT BETKE, RONALD L. RIVEST, MONA SINGH. Piecemeal Learning of an Unknown Environment. In *Sixth ACM Conference on Computational Learning Theory (COLT 93)*, pages 277–286, July 1993.

[2] K-F. CHAN, T. W. LAM. An on-line algorithm for navigating in an unknown environment. *International Journal of Computational Geometry & Applications*, 3:227–244, 1993.

[3] W. CHIN, S. NTAFOS. Optimum Watchman Routes. *Information Processing Letters*, 28:39–44, 1988.

[4] W. CHIN, S. NTAFOS. Shortest Watchman Routes in Simple Polygons. *Discrete and Computational Geometry*, 6(1):9–31, 1991.

[5] A. DATTA, CH. HIPKE, S. SCHUIERER. Competitive Searching in Polygons—Beyond Generalized Streets. In *Proc. Sixth Annual International Symposium on Algorithms and Computation*, pages 32–41. LNCS 1004, 1995.

[6] A. DATTA, CH. ICKING. Competitive Searching in a Generalized Street. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, pages 175–182, 1994.

[7] X. DENG, T. KAMEDA, C.H. PAPADIMITRIOU. How to Learn an Unknown Environment. In *Proceedings of Foundations of Computer Science*, pages 298–303, 1991.

[8] F. Hoffmann, C. Icking, R. Klein, K. Kriegel. A Competitive Algorithm for Learning a Polygon. In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 166–174, 1997.

[9] Christian Icking, Rolf Klein. Searching for the Kernel of a Polygon: A Competitive Strategy. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 258–266, 1995.

[10] R. Klein. Walking an unknown street with bounded detour. *Comput. Geom. Theory Appl.*, 1:325–351, 1992.

[11] J. M. Kleinberg. On-line search in a simple polygon. In *Proc. of 5th ACM-SIAM Symp. on Discrete Algorithms*, pages 8–15, 1994.

[12] A. López-Ortiz, S. Schuierer. Going Home Through an Unknown Street. In S. G. Akl, F. Dehne, J.-R. Sack, editors, *Proc. 4th Workshop on Algorithms and Data Structures*, pages 135–146. LNCS 955, 1995.

[13] Aohan Mei, Yoshihide Igarashi. An Efficient Strategy for Robot Navigation in Unknown Environment. *Inform. Process. Lett.*, 52:51–56, 1994.

[14] C. H. Papadimitriou, M. Yannakakis. Shortest paths without a map. In *Proc. 16th Internat. Colloq. Automata Lang. Program.*, volume 372 of *Lecture Notes in Computer Science*, pages 610–620. Springer-Verlag, 1989.

[15] D. D. Sleator, R. E. Tarjan. Amortized efficiency of list update and paging rules. *Communications of the ACM*, 28:202–208, 1985.